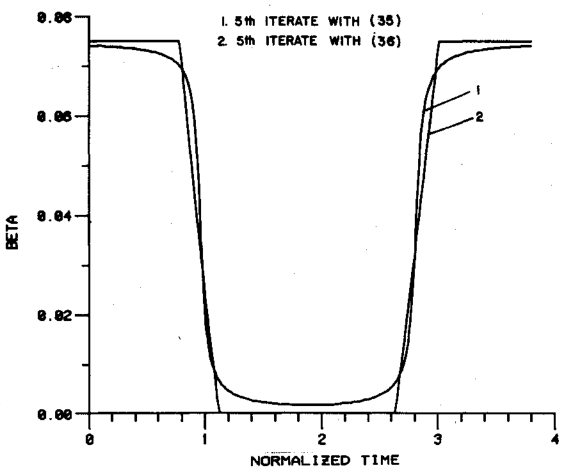

Fig. 2 Time history of control $\beta(t)$.

The Jacobian is discretized using Eq. (4) with $h=.000001$. We remark that h could be increased if needed. The initial function $x^{(0)}(t)$ is essentially the same as that in Ref. 1 except for $x_8(t)$, which is taken as 1 on the interval.

By using the approximations (35) and (36), we employ only 8 state variables, whereas the procedure in Ref. 1 employs 11 by converting the inequality constraint (25) into an equality constraint.

Let n denote the number of subintervals into which [0,3.186] is divided, i.e., $\Delta t=3.816/n$. We performed numerical experiments with the following parameters:

Experiment 1: $\alpha=20$ in Eq. (35), $n=200$

Experiment 2: $\alpha=20$ in Eq. (35), $n=40$

Experiment 3: $k=5$ in Eq. (36), $n=200$

Numerical experiments with $\alpha=25$ in Eq. (35) became unstable even when Δt is increased. On the other hand, with $\alpha=20$, increasing Δt did not affect the stability of the procedure as experiment 3 indicates. Increasing k to 7 in Eq. (36) made the procedure unstable with $n=200$. A better choice of $x^{(0)}(t)$ would of course increase the limiting values of α and k.

With experiment 1 the sequence $\{x^{(p)}\}$ converges to an accuracy of 4 significant digits in 5 iterations, and with experiment 2 a 4-digit accuracy was achieved at the seventh iterate. With experiment 3 the sequence converges to an accuracy of 4 significant digits in 5 iterations, and the fifth iterate has

$$\psi_1(0)=0.8794185E+00 \qquad \psi_2(0)=0.4203096E+00$$

$$\psi_3(0)=0.1027116E+01 \qquad \psi_4(0)=0.7651173E+00 \tag{37}$$

Figures 1 and 2 give plots of $\theta(t)$ vs t and $\beta(t)$ vs t for the fifth iterates of experiments 1 and 3. These compare very well with the exact plots given in Ref. 1. The values of θ and β from experiment 2 are close to those of experiment 1 and are not plotted. We note that the plots of β in Fig. 2 approximate the bang-bang control better than those given in Ref. 1.

To get an idea of the relative computing times involved, experiment 1 took 7 min and 30 s of execution time (VAX 11/730) to compute 5 iterates, whereas experiment 3 required 7 min and 50 s for 5 iterates.

Table 1 gives the output of experiments 1 and 3, and it can be observed that the responses are close.

References

[1]Kenneth, P. and McGill, R., "Two-Point Boundary Value Problem Techniques," *Advances in Control Systems,* Vol. 3, edited by C.T. Leondes, Academic Press, New York, 1966, pp. 69-110.

[2]Bellman, R. and Kalaba, R., *Quasilinearization and Nonlinear Boundary Value Problems,* American Elsevier, New York, 1965, pp. 75-84.

[3]Kalaba, R., "On Nonlinear Differential Equations: The Maximum Operation and Montone Convergence," *Journal of Mathematics and Mechanics,* Vol. 8, 1959, pp. 519-574.

[4]McGill, R. and Kenneth, P., "A Convergence Theorem on the Iterative Solution of Nonlinear Two Point Boundary Value Systems," *Proceedings of the XIV International Astronautical Congress,* Paris, 1963, pp. 173-186.

[5]Subrahmanyam, M.B., "An Algorithm for the Solution of Nonlinear Multi-point Boundary Value Problems by Quasilinearization," submitted for publication.

[6]Pontryagin, L.S., Boltyanski, V.G., Gamkrelidze, R.V., and Mischenko, E.F., *The Mathematical Theory of Optimal Processes,* Interscience Publishers, New York, 1962, pp. 9-73.

[7]Greensite, A.L., *Elements of Modern Control Theory,* Vol. 1, Spartan Books, New York, 1970, pp. 586-595.

Near-Optimum Design of Large-Scale Systems

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Introduction

A PROPOSED scheme of decentralized near-optimization is applied to linear, time-invariant, interconnected large-scale systems. Due to the characteristics of such systems, the optimal control designs are, for the most part, necessarily near-optimum.[1-3] Several authors have applied different methods for near-optimum design of large-scale systems.[2] In the decentralized methods[1-2] the multilevel near-optimal controller design is based on minimizing the effect of interconnections, without reference to their possible beneficial roles. In this paper we propose a new scheme of decentralized near-optimum design based on using a multilevel feedback controller. It is assumed that the large-scale system is decomposed into ℓ subsystems, which are optimized by local feedback controllers with respect to a local performance index while ignoring the interactions among the subsystems. Then, the design of the global controller is performed using the available information on the subsystem level. To evaluate the near-optimization scheme, we compare it with that given by Siljak in Ref. 1, through applying both methods to a numerical example.

Near-Optimal Design

Consider the linear time-invariant interconnected system in the input-decentralized form[1]

$$\dot{x}_i=A_ix_i+b_iu_i+\sum_{\substack{j=1\\j\neq i}}^{\ell}N_{ij}x_j, \quad i=1,\ldots,\ell \tag{1}$$

where $x_i \epsilon R^{n_i}$ and $u_i \epsilon R^{m_i}$. We assume that $x=0$ is the only equilibrium solution of Eq. (1).

In the proposed multilevel near-optimization scheme, the control $u_i(t)$ is considered as consisting of two parts, the local

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control $u_i^\ell(t)$ and the global control $u_i^g(t)$, such that

$$u_i(t) = u_i^\ell(t) + u_i^g(t) \tag{2}$$

The local control $u_i^\ell(t)$ is chosen as a linear control law

$$u_i^\ell(t) = -k_i^\ell x_i(t) \tag{3}$$

to optimize the isolated subsystem, and the global control law

$$u_i^g(t) = -\sum_{\substack{j=1\\ j\neq i}}^{\ell} k_{ij}^g x_j(t) \tag{4}$$

is chosen to satisfy the near-optimization of the performance of the large-scale interconnected system.

Now, for designing the local control $u_i^\ell(t)$, Eq. (1) may be viewed as the interconnection of ℓ free (isolated) subsystems described by the equation

$$\dot{x}_i = A_i x_i + b_i u_i^\ell, \qquad i = 1,\ldots,\ell \tag{5}$$

We assume that all ℓ pairs (A_i, b_i) are controllable and that, with each isolated subsystem given by Eq. (5), the quadratic performance index[2] is

$$J_i(u_i^\ell) = \tfrac{1}{2}\int_0^\infty [x_i^T Q_i x_i + (u_i^\ell)^T R_i u_i^\ell]\,\mathrm{d}t \tag{6}$$

where Q_i is a symmetric nonnegative definite matrix and R_i is a symmetric positive definite matrix.

The local control u_i^ℓ can now be chosen to minimize the performance index $J_i(u_i^\ell)$ in Eq. (6). Following the maximum principle approach for solving the linear regulator,[4] it follows that the Hamiltonian is

$$H_i^\ell = \tfrac{1}{2} x_i^T Q_i x_i + \tfrac{1}{2}(u_i^\ell)^T R_i u_i^\ell + (\lambda_i^\ell)^T (A_i x_i + b_i u_i^\ell) \tag{7}$$

From Eq. (7) we have

$$u_i^\ell(t) = -R_i^{-1} b_i^T \lambda_i^\ell(t) \tag{8}$$

Assume that the solution for the adjoint $\lambda_i^\ell(t)$ is

$$\lambda_i^\ell(t) = P_i x_i(t) \tag{9}$$

where P_i is a time-invariant $n_i \times n_i$ symmetric, positive definite matrix that satisfies the $n_i(n_i+1)/2$ algebraic equation

$$Q_i + P_i A_i + A_i^T P_i - P_i b_i R_i^{-1} b_i^T P_i = 0 \tag{10}$$

From Eqs. (8) and (9), it follows that

$$u_i^\ell(t) = -R_i^{-1} b_i^T P_i x_i(t) \tag{11}$$

Now, for the interconnected system given by Eq. (1) to be nearly optimized with respect to the quadratic performance index,

$$J(u) = \sum_{i=1}^{\ell} J_i(u_i) = \sum_{i=1}^{\ell} \int_0^\infty \tfrac{1}{2}[x_i^T Q_i x_i + u_i^T R_i u_i]\,\mathrm{d}t \tag{12}$$

The Hamiltonian will be

$$H = \sum_{i=1}^{\ell} \left\{ \tfrac{1}{2} x_i^T Q_i x_i + \tfrac{1}{2} u_i^T R_i u_i + \lambda_i^T \Big(A_i x_i + b_i u_i + \sum_{\substack{j=1\\ j\neq i}}^{\ell} N_{ij} x_j\Big) \right\} \tag{13}$$

Equation (13) can be rewritten as

$$H = \sum_{i=1}^{\ell} H_i$$

where

$$H_i = \tfrac{1}{2} x_i^T Q_i x_i + \tfrac{1}{2} u_i^T R_i u_i + \lambda_i^T (A_i x_i + b_i u_i) + \sum_{\substack{j=1\\ j\neq i}}^{\ell} \lambda_j^T N_{ji} x_i \tag{14}$$

To find the control u_i, which minimizes the Hamiltonian H_i, we have

$$\frac{\partial H_i}{\partial u_i} = 0 = R_i u_i + b_i^T \lambda_i \tag{15}$$

and

$$\frac{\partial H_i}{\partial x_i} = -\dot{\lambda}_i = Q_i x_i + A_i^T \lambda_i + \sum_{\substack{j=1\\ j\neq i}}^{\ell} N_{ji}^t \lambda_j \tag{16}$$

From Eq. (15) we have

$$u_i(t) = -R_i^{-1} b_i^T \lambda_i(t) \tag{17}$$

At this point we assume that the solution for the adjoint $\lambda_i(t)$ is

$$\lambda_i(t) = P_i x_i(t) - \xi_i(t) \tag{18}$$

In this case $\xi_i(t)$ represents the part of the adjoint $\lambda_i(t)$ that reflects the effect of interconnection structure given by N_{ij}. From Eqs. (1) and (16-18), it follows that

$$\dot{\xi}_i = (P_i b_i R_i^{-1} b_i^T - A_i^T)\xi_i - \sum_{\substack{j=1\\ j\neq i}}^{\ell} N_{ji}^T \xi_j + \sum_{\substack{j=1\\ j\neq i}}^{\ell} (P_i N_{ij} + N_{ji}^T P_j) x_j \tag{19}$$

From Eqs. (2), (4), (11), (17), and (18), it follows that

$$u_i^g = R_i^{-1} b_i^T \xi_i = \sum_{\substack{j=1\\ j\neq i}}^{\ell} k_{ij}^g x_j \tag{20}$$

An approximate form of Eq. (19) was chosen as

$$\dot{\xi}_i = (P_i A_i - P_i b_i R_i^{-1} b_i^T P_i) P_i^{-1} \xi_i - \sum_{\substack{j=1\\ j\neq i}}^{\ell} N_{ji}^T \xi_j \tag{21}$$

Comparing Eqs. (19) and (21) and using Eq. (10), we have

$$\xi_i = -P_i (Q_i + P_i b_i R_i^{-1} b_i^T P_i)^{-1} \sum_{\substack{j=1\\ j\neq i}}^{\ell} (P_i N_{ij} + N_{ji}^T P_j) x_j \tag{22}$$

and thus

$$u_i^g(t) = -R_i^{-1} b_i^T P_i (Q_i + P_i b_i R_i^{-1} b_i^T P_i)^{-1} \sum_{\substack{j=1\\ j\neq i}}^{\ell} (P_i N_{ij} + N_{ji}^T P_j) x_j(t)$$

Note that the matrix $(Q_i + P_i b_i R_i^{-1} b_i^T P_i)^{-1}$ exists since the free subsystems are optimally stabilized. From Eqs. (17), (18) and (22) it follows that

$$u_i = -R_i^{-1} b_i^T P_i x_i - R_i^{-1} b_i^T P_i (Q_i + P_i b_i R_i^{-1} b_i^T P_i)^{-1} \sum_{\substack{j=1\\ j\neq i}}^{\ell} (P_i N_{ji} + N_{ji}^T P_j) x_j \tag{23}$$

where

$$K_i^\ell = -R_i^{-1} b_i^T P_i \tag{24}$$

and

$$K_{ij}^g = -R_i^{-1} b_i^T P_i (Q_i + P_i b_i R_i^{-1} b_i^T P_i)^{-1} (P_i N_{ij} + N_{ji}^T P_j) \tag{25}$$

To evaluate our proposed scheme of near-optimization, we applied it to several examples[5] and compared the results with those given by Siljak.[1] Both methods were compared with the exact optimal control solution. The following example demonstrates such comparison.

Numerical Example

Consider the linear system, where the overall system consists of two linearly interconnected, strongly coupled subsystems represented by

$$\dot{x}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1, \quad x_1 = (z_1, z_2)^T$$

$$\dot{x}_2 = 5x_2 + [1 \quad 4] x_1 + u_2, \quad x_2 = z_3 \tag{26}$$

We wish to find the control vector $u = (u_1, u_2)^T$ that minimizes

$$J(u) = \sum_{i=1}^{2} J_i(u_i)$$

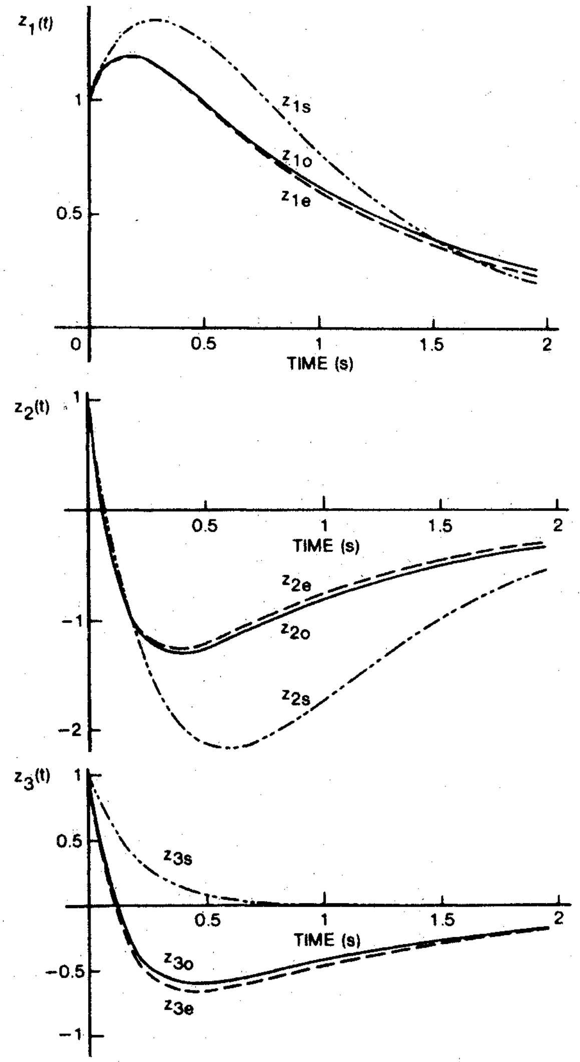

Fig. 1 Results of numerical example.

where

$$J_i(u_i) = \frac{1}{2} \int_0^\infty \{x_i^T Q_i x_i + u_i^T R_i u_i\} \mathrm{d}t, \quad i = 1,2$$

where

$$Q_1 = \text{unit matrix}, \quad Q_2 = R_1 = R_2 = 1$$

The pairs (A_i, b_i), $i = 1,2$ are controllable. From Eq. (23) we have

$$u_{1_o} = [-8.424181 \quad -8.181541] \; x_{1_o} - 8.106149 \, x_{2_o}$$

$$u_{2_o} = -10.099020 \; x_{2_o} - [6.898873 \;\; 6.391752] \; x_{1_o} \tag{27}$$

where o denotes our near-optimization approach. Now, applying Siljak's algorithm in Ref. 1, the control will be

$$u_{1s} = [-8.424181 \quad -8.181541] \; x_{1s} \quad -2x_{2s}$$

$$u_{2s} = -10.099021 \;\; x_{2s} - [1 \quad 4] \; x_{1s} \tag{28}$$

where s refers to Siljak's method.

Now, applying the "one-shot" exact optimization to system (26), the optimal control for the overall system is

$$u_{1e} = [-8.744568 \quad -8.932045] \; x_{1e} - 6.880953 \; x_{2e}$$

$$u_{2e} = -9.582484 \;\; x_{2e} - [7.411487 \; 6.880955] \; x_{1e} \tag{29}$$

where e refers to exact optimization.

To check the exact system's optimal control solution vs the Siljak near-optimal control solution and our proposed near-optimal control solution, system (26) was solved using the controls given in Eqs. (27-29) with $x(0) = 1 \;\; 1 \;\; 1)^T$. The results are shown in Fig. 1 for the time interval $0 \leq t \leq 2.0$. Figure 1 indicates that the near-optimal closed-loop control considered here is a good approximation when compared with Siljak's near-optimal closed-loop control.

Conclusions

In this proposed scheme of near-optimization, each subsystem is nearly optimized with respect to its own performance index using both local and global feedback controllers. Our proposed scheme enables us nearly to optimize the system on the lower-order subsystem level. This is important for large-scale interconnected systems because, if the system is too large, the one-shot optimization of the system is either uneconomical, requiring excessive computer time, or impossible, demanding excessive computer storage to complete the optimization.

In this near-optimal scheme, we assumed a beneficial role for the interconnections. This offers an advantage over Siljak's algorithm, in which the role of the global controls is to reduce the effect of interconnections, without reference to their possible beneficial effects.

References

[1]Siljak, D.D., *Large-Scale Dynamic Systems: Stability and Structure,* Elsevier North-Holland, Inc., New York, 1978.

[2]Jamshidi, M., *Large-Scale Systems: Modeling and Control,* Elsevier North-Holland, Inc., New York, 1983.

[3]Michel, A.N. and Miller, R.K., *Qualitative Analysis of Large-Scale Dynamical Systems,* Academic Press, Orlando, FL, 1977.

[4]Sage, A.P. and White, C.C., *Optimum Systems Control,* Prentice-Hall, Inc., Englewood Cliffs, NJ, 1977.

[5]Matta, N.R., "Stability and Control of Large-Scale Interconnected Systems,' Ph.D. Thesis, George Washington University, Washington, DC, April 1985.